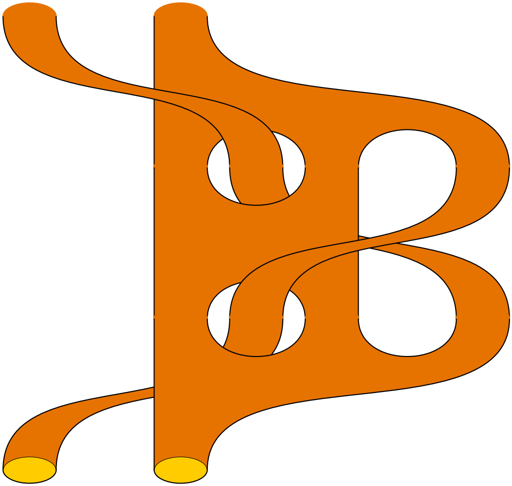
YTM 2024: Segal Infinity Props



Kurt Stoeckl joint with Philip Hackney and Marcy Robertson

Props

Let $\mathfrak{C}$ be a set of colours, and denote a sequence of colours $\underline{c} = (c_1, ..., c_n)$

Definition

A $\mathfrak{C}$-coloured **prop** P is a strict symmetric monoidal category whose objects are generated by the free monoid $F(\mathfrak{C})$.

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Let $P\binom{\underline{d}}{\underline{c}} := Hom_P(c_1 \otimes \dots \otimes c_n, d_1 \otimes \dots \otimes d_m)$, then

- P is a (symmetric) bimodule, i.e. if $\alpha \in P\binom{\underline{d}}{\underline{c}}$ then $\alpha \cdot \binom{\sigma}{\tau} \in P\binom{\sigma \underline{d}}{\underline{c}\tau}$.
- P has a vertical composition $P\binom{\underline{c}}{\underline{b}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\circ_V} P\binom{\underline{c}}{\underline{a}}$.
- P has a horizontal composition $P\binom{\underline{d}}{\underline{c}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\circ_h} P\binom{\underline{d}, \underline{b}}{\underline{c}, \underline{a}}$.

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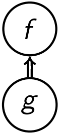
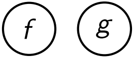
Denote $\circ_V(f, g)$ with

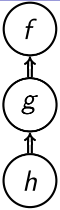


and, $\circ_h(f, g)$ with



The Relations of a Prop

$$\circ_V(\circ_V(f,g),h) = \circ_V(f,\circ_V(g,h))$$



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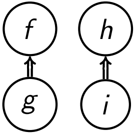

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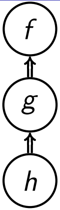

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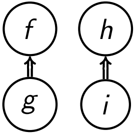

Idea: An ∞-prop, is a prop where,

1. these relations no longer hold strictly*, but only up to homotopy, and
2. we can construct an infinity tower of homotopies between homotopies.

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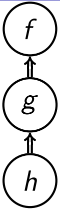

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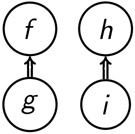


Idea: An ∞-prop, is a prop where,

1. these relations no longer hold strictly*, but only up to homotopy, and
2. we can construct an infinity tower of homotopies between homotopies.

*:Props also have unit and biequivariance relations which we are not relaxing today.

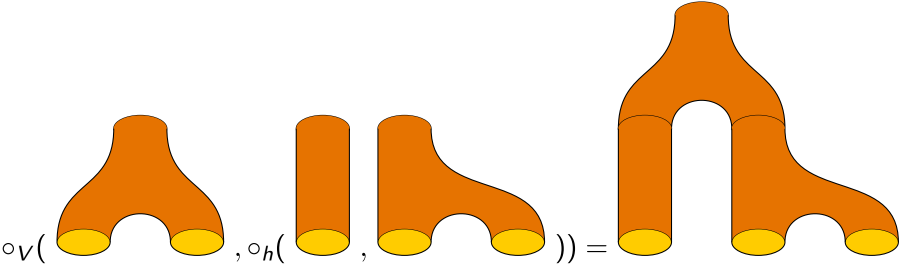
Oriented Cobordisms

Theorem (Godin, Santander)

Normalised fat admissible graphs provides a model for the classifying spaces of mapping class groups of oriented cobordisms.

Theorem (Santander)

They also provide a model for the composition of oriented cobordisms.



Is this composition governed by a prop?

Path Spaces

If X is a topological space, then in $P(X) := Hom_{Top}([0,1], X)$ one can compose paths with the same target and source,

$$\circ_V(f,g) := \begin{cases} g(2t), & 0 \leq t \leq 1/2 \\ f(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

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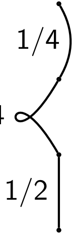
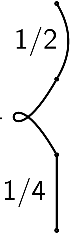
However, as this composition involves scaling it is not strictly associative...

$\circ_V(\circ_V(f,g),h) = 1/4$



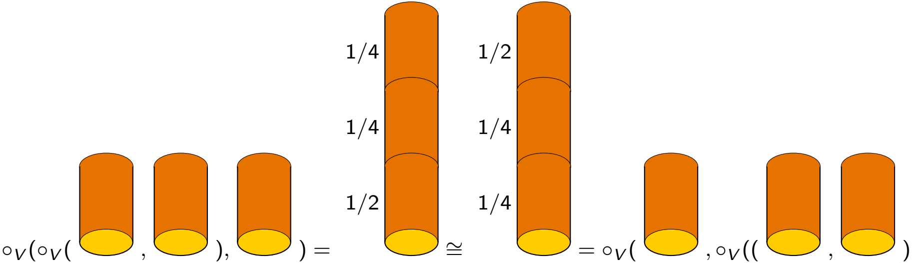
$\cong 1/4$



$= \circ_V(f, \circ_V(g,h))$

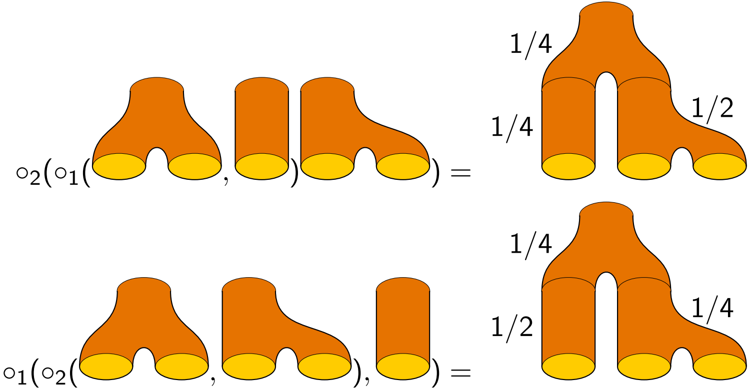
Cobordisms Have ∞-Category Homotopies

The composition of fat graphs also involves normalisation, thus informally*

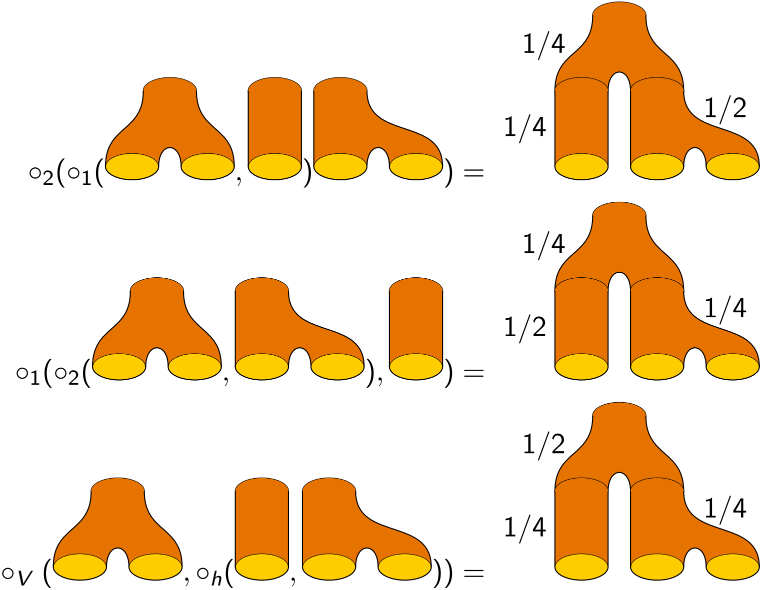


* : We are illustrating the scalings on the cobordisms, not actual fat graphs.
The true compositions + scalings of fat graphs are much more complicated!

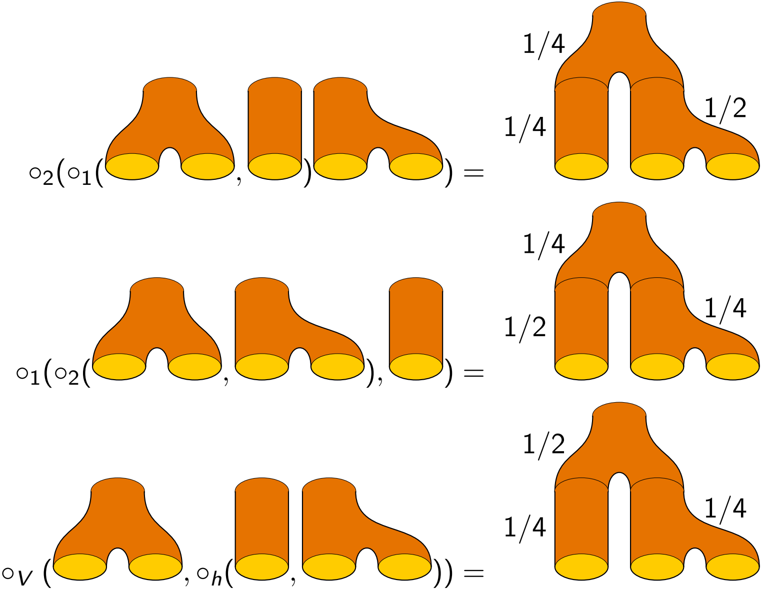
Cobordisms Should Have ∞-Operad Homotopies,



Cobordisms Should Have ∞-Operad Homotopies, and ...



Are These Homotopies Governed by an ∞-Prop?



But what is an ∞-prop?

A Simplicial Reminiscence

Definition

The simplicial category $\triangle$ has

- as objects, the totally ordered sets $[n] := \{0 < 1 < ... < n\}$, and
- as morphisms, the order preserving functions between them.

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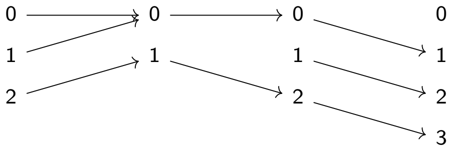
Theorem

Every morphism $f : [n] \to [k]$ in $\triangle$ admits a unique factorisation up to unique isomorphism,

$$\begin{array}{ccc} [n] & \xrightarrow{f} & [k] \\ \sigma \downarrow & & \uparrow h \\ [n'] & \xrightarrow[\delta]{} & [k'] \end{array}$$

- *σ is a composite of codegeneracy maps.*
- *δ is a composite of inner coface maps.*
- *h is a composite of outer coface maps.*



Simplicial Sets

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Example

Let $\mathcal{C}$ be a small category, then the nerve $N\mathcal{C}$ is the simplicial set with

- $(N\mathcal{C})([0])$ = the objects of $\mathcal{C}$.
- $(N\mathcal{C})([1])$ = the morphisms of $\mathcal{C}$.
- $(N\mathcal{C})([n])$ = the strings of n-composable morphisms of $\mathcal{C}$.

The simplices are related by,

- degeneracy maps, $s_i(\xrightarrow{f_1} \dots \xrightarrow{f_n}) = \xrightarrow{f_1} \dots \xrightarrow{f_i}\xrightarrow{id}\xrightarrow{f_{i+1}} \dots \xrightarrow{f_n})$
- inner face maps, $d_i(\xrightarrow{f_1} \dots \xrightarrow{f_n}) = \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}}\xrightarrow{f_{i+1}f_i}\xrightarrow{f_{i+2}} \dots \xrightarrow{f_n})$
- outer face maps, $d_0(\xrightarrow{f_1} \dots \xrightarrow{f_n}) = \xrightarrow{f_2} \dots \xrightarrow{f_n}$ and $d_n(\xrightarrow{f_1} \dots \xrightarrow{f_n}) = \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}}$.

Relation to Categories

Definition (The Nerve)

The functor $N : Cat \to Set^{\triangle^{op}}$ is defined to send each $\mathcal{C} \in Cat$, to

$$(N\mathcal{C})([n]) := Cat([n], \mathcal{C})$$

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for all $[n] \in \triangle$.

Theorem

The nerve functor $N : Cat \to Set^{\triangle^{op}}$ is fully faithful. Moreover, for every simplex $X \in Set^{\triangle^{op}}$, the following statements are equivalent.

- *There exists a small category $\mathcal{C}$ and an isomorphism $X \cong N\mathcal{C}$.*
- *X satisfies the strict Segal condition.*
- *X satisfies the strict inner Kan condition.*

In other words, we have equivalences of categories,

$$Cat \cong Set^{\triangle^{op}}_{Segal} \cong Set^{\triangle^{op}}_{Kan}$$

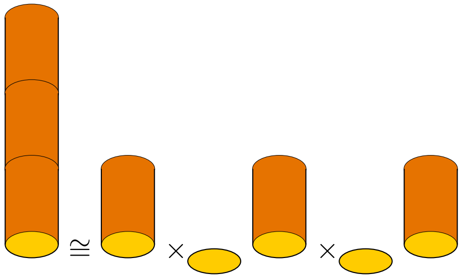
What Are These Conditions Saying?

Definition

A simplicial set X is said to satisfy the **Segal condition** if we have a weak homotopy equivalence,

$$X([n]) \cong X([1]) \times_{X([0])} \dots \times_{X([0])} X([1])$$

If this is an isomorphism, we say X satisfies the **strict Segal condition**.



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Let $\triangle[n]$ denote the **representable presheaf** at $[n]$, i.e.

$$\triangle[n] := Hom_{\triangle}(-, [n])$$

Let $\delta : [n-1] \to [n]$ be a inner coface map, then $\Lambda^{\delta}[n] \subset \triangle[n]$ is called the **inner horn** of δ, and consists of all coface maps into $[n]$ except δ.

Definition

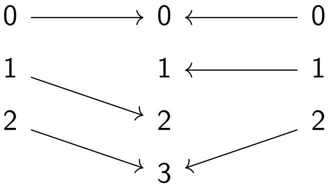
A simplicial set X is said to be an ∞-category if

$$\begin{array}{ccc} \Lambda^{\delta}[n] & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \triangle[n] & & \end{array}$$

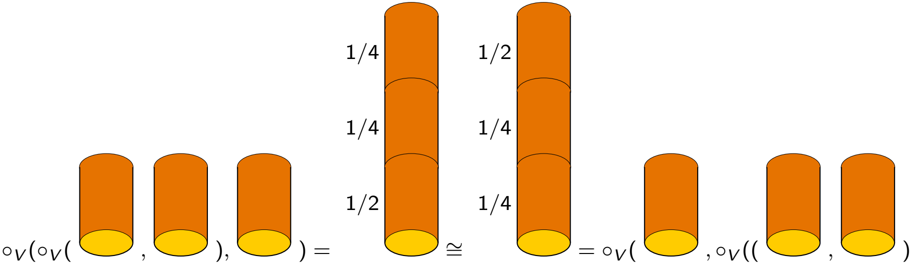
admits a filler. It is said to be **strict inner Kan** if the filler is unique.

What Are These Conditions Saying?

There are only two inner coface maps targeting [3],



Filling in these two inner horns lets us construct the homotopy



Do These Approaches Differ?

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Theorem (Joyal and Tierney)

There exist two Quillen equivalent models for ∞-categories,

1. *The Rezk model on complete Segal spaces* $sSet^{\triangle^{op}}$
2. *The Joyal model on simplicial sets* $Set^{\triangle^{op}}$

There are many others!

∞-Props

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Theorem (Hackney, Robertson, S.)

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Theorem (Hackney, Robertson, S.)

There exist two models for ∞-Props,

1. *An analogue of the Rezk model, on complete graphical spaces $sSet^{\Gamma^{op}}$*
2. *An analogue of the Joyal model, on graphical sets $Set^{\Gamma^{op}}$*

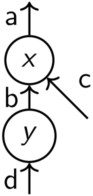
We are currently working to show they are Quillen equivalent.

An Analogue of the Simplicial Category

Definition (Kock)

A directed graph G is a diagram of finite sets,

$$Edges \hookleftarrow Inputs \longrightarrow Vertices \longleftarrow Outputs \hookrightarrow Edges$$

$G =$ 

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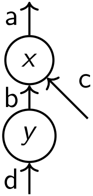
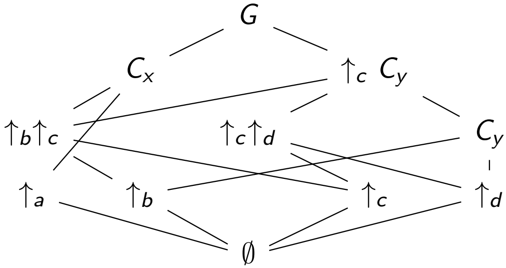
Let $Sb(G)$ denote the set of subgraphs of G. Where a subgraph is a subset of $2^{E \times V}$ which is convex closed and each vertex has all inputs+outputs.

$G =$ 

, $Sb(G) =$ 

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The graphical category Γ has as objects the directed **acyclic** graphs. If G and K are graphs, a morphism $f : G \to K$ is

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- such that if $H \in Sb(G)$ then $f(H) \in Sb(K)$.

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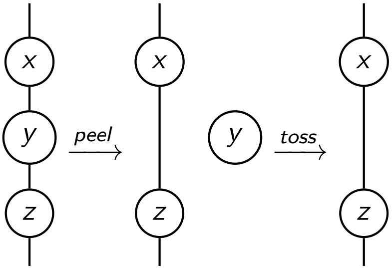
Every morphism $f : G \to K$ in Γ admits a unique factorisation up to unique isomorphism,

$$\begin{array}{ccc} G & \xrightarrow{f} & K \\ \sigma\downarrow & & \uparrow h \\ G_1 & \xrightarrow[\delta]{} & G_3 \end{array}$$

- *σ is a composite of codegeneracy maps.*
- *δ is a composite of inner coface maps.*
- *h is a composite of outer coface maps.*

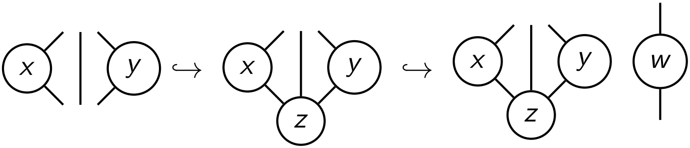
Example Morphisms

The **codegeneracy maps** peel off edges, or toss away nullary vertices, e.g.



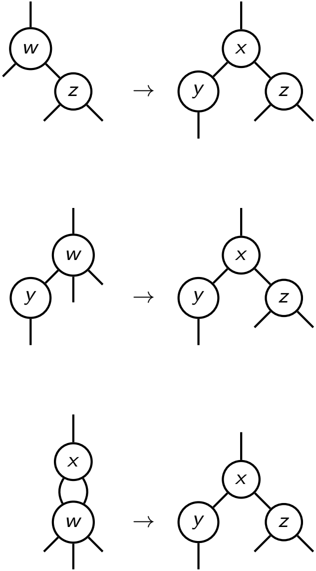
Example Morphisms

The **outer coface maps** correspond to subgraph inclusions $G \hookrightarrow K$, where K has one additional vertex, e.g.

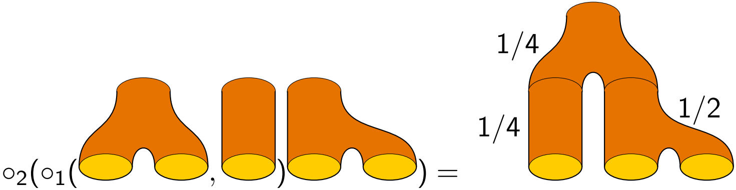


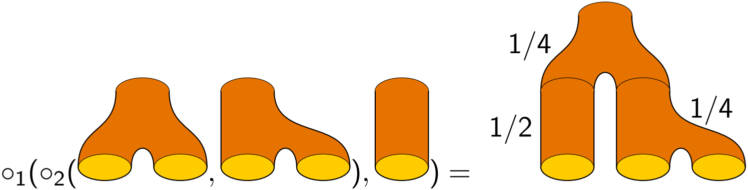
Example Morphisms

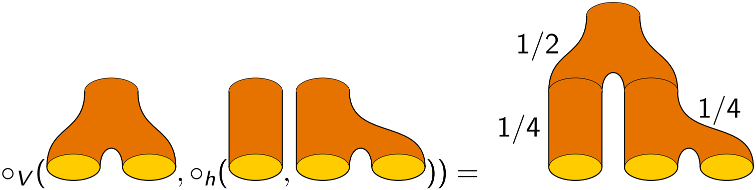
The **inner coface maps**, tear apart vertices, either producing internal edges, or empty space, e.g.



Filling in the Horns Yields the Homotopies







Are the normalised fat graphs of Santander an ∞-Prop?

Are the normalised fat graphs of Santander an ∞-Prop?

Two weeks ago with Luci Bonatto:

All homotopies of the prior slide hold strictly in normalised fat graphs.

More generally, we expect.

Conjecture

The normalised fat graphs of Santander are a type of ∞-prop. In particular, it is the envelope of an ∞-properad.

Much of the ∞-properadic/operadic structure is actually strict as well!

Mentioned Sources

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