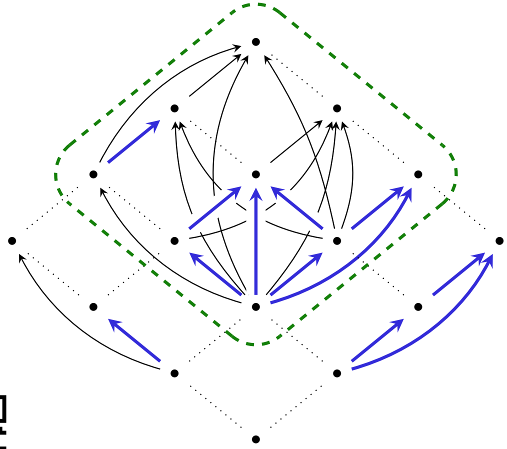
Maximal Compatibility of G-Transfer Systems



David DeMark, Mike Hill, Yigal Kamel, Nelson Niu,
Kurt Stoeckl, Danika Van Niel and Guoqi Yan

Some Motivation

Definition

A commutative monoid is a set R, with a map $\mu : R \times R \to R$ which is

- commutative: $\mu(x, y) = \mu(y, x)$, and
- associative: $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$.

Unfortunately topologists like sets with topology, such as

- Topological spaces, and
- Topological G-spaces.

Asking for strict commutativity and associativity is too much...

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...so we relax both conditions up to homotopy...

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- Topological G-spaces (and G-spectra) have many!
  - For every non-trivial G, there are many distinct G-N_∞-operads.

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- Topological G-spaces (and G-spectra) have many!
  - For every non-trivial G, there are many distinct G-N_∞-operads.

Theorem ([BH15, GW18, BP21, Rub21a, BMO24])

There is an equivalence of categories $Ho(G\text{-}N_\infty\text{-}Operads) \cong Tr(G)$.

Nasty homotopy commutative monoids in G-spaces

$\cong$

Nice combinatorics of **G-transfer systems**

G-Transfer Systems

Definition

Let $\mathcal{O}$ be a binary relation on $Sub(G)$ refining $\subseteq$. Then, $\mathcal{O}$ is said to be a G-transfer system if it is closed under

- conjugation,
- restriction, and
- composition.

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$$K \to H \quad \overset{\forall g \in G}{\Longrightarrow} \quad gKg^{-1} \to gHg^{-1}$$

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& H \arrow[dr, dash, dotted, "\supseteq"] & \\
K \arrow[ur, red] & & J
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
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& K \cap J \arrow[ur, red] &
\end{tikzcd}
$$

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Special case: $K \subseteq J$

$$\begin{tikzcd}
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H \arrow[dr, dash, dotted] & \\
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- composition.

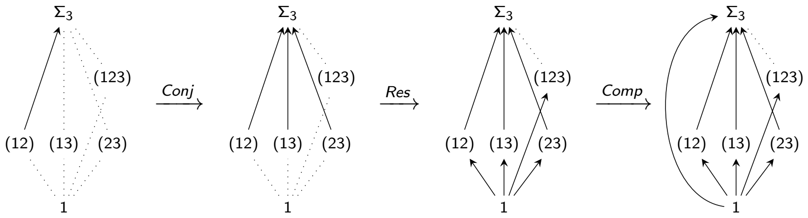
Theorem (A.2 of [Rub21a])

Let R be a binary relation on $Sub(G)$ refining $\subseteq$. Let $T(R)$ denote the closure of R under

- *conjugation, then*
- *restriction, and then*
- *composition.*

Then $T(R)$ is the smallest G-transfer system containing R.

Example



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Compatible Transfer Systems

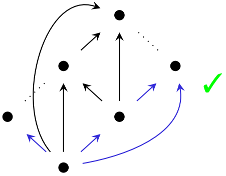
Definition ([Cha24, Definition 4.6])

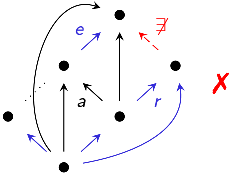
Let $\mathcal{O}$ and $\mathcal{O}_m$ be a pair of G-transfer systems such that $\mathcal{O}_m \subseteq \mathcal{O}$. We say $(\mathcal{O}, \mathcal{O}_m)$ are **compatible** if we can complete all squares of the form

$$\begin{tikzcd} & H & \\ K \arrow[ur, "e"] & & J \arrow[ul, dashed, "\exists a'"'] \\ & K \cap J \arrow[ul, "a"] \arrow[ur, "r"'] & \end{tikzcd}$$

i.e. if $e \in \mathcal{O}_m, r \in \mathcal{O}_m$ and $a \in \mathcal{O}$ then we must have $a' \in \mathcal{O}$.

A C_{p^2q} example (left), and counter example (right).



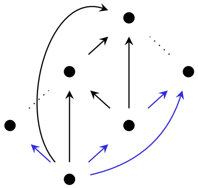


Maximal Compatible Transfer Systems

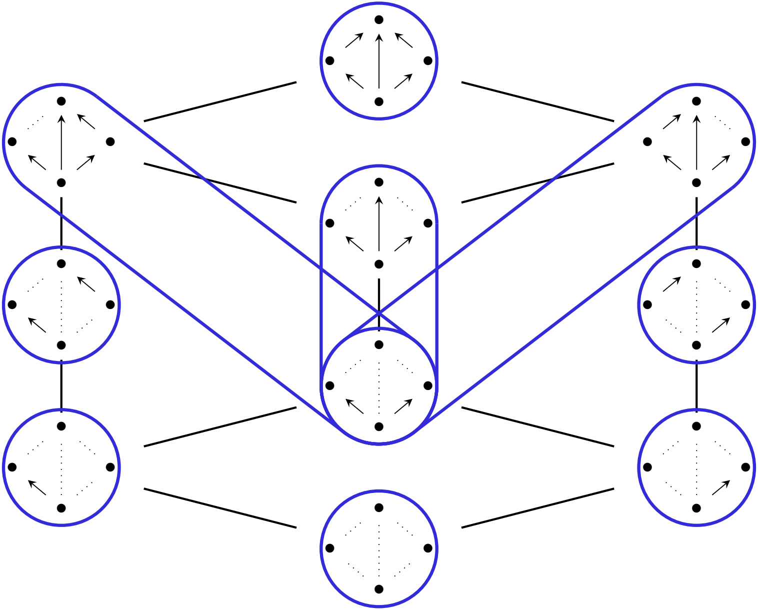
Proposition ([BH22])

For a fixed transfer system $\mathcal{O}$,

- *there exists a maximal compatible transfer system $\mathcal{M}(\mathcal{O})$, and*
- *$(\mathcal{O}, \mathcal{O}_m)$ are compatible, if, and only if, $\mathcal{O}_m \subseteq \mathcal{M}(\mathcal{O})$.*



Maximal Compatible Pairs of C_{pq}-Transfer Systems



But Why Care About Maximal Compatibility?

Theorem ([BH15, GW18, BP21, Rub21a, BMO24])

There is an equivalence of categories $Ho(G\text{-}N_\infty\text{-}Operads) \cong Tr(G)$.

Compatible additive and multiplicative ring structures in G-spaces

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Compatible transfer systems

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Compatible additive and multiplicative ring structures in G-spaces

$\cong$

Compatible transfer systems

Determine all multiplicative ring structures that distribute over an addition

$\cong$

Compute the maximal compatible transfer system

Computing $\mathcal{M}(\mathcal{O})$

What if we just delete everything that produces a compatibility failure?

Computing $\mathcal{M}(\mathcal{O})$ via its Complement

What if we just delete everything that produces a compatibility failure?

Proposition (DHKNSVNY)

The complement $\mathcal{M}(\mathcal{O})^c := \mathcal{O} \setminus \mathcal{M}(\mathcal{O})$ of the maximal compatible transfer system of $\mathcal{O}$ satisfies,

$$\mathcal{M}(\mathcal{O})^c = \{e \in \mathcal{O} : \exists r, r' \in Res(e), a \in \mathcal{O}, \nexists a' \in \mathcal{O} \textit{ such that}$$

$$\begin{tikzcd}
 & & H & \\
 & J \arrow[ur, dotted, no head] & & K \arrow[ul, "e"'] \\
I \arrow[ur, dashed, red, "\nexists a'"] & & K \cap J \arrow[ul, "r"'] \arrow[ur, dotted, no head] & \\
 & K \cap I \arrow[u] \arrow[ul, "r'"] \arrow[ur, "a"'] & &
\end{tikzcd} \}.$$

Whenever this diagram occurs we delete every bold arrow.

Case: $\mathcal{M}(\mathcal{O})^c = \emptyset$, i.e. $\mathcal{O}$ is Self-Compatible

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Corollary (DHKNSVNY)

A transfer system $\mathcal{O}$ is self compatible, if, and only if, it is saturated, i.e.

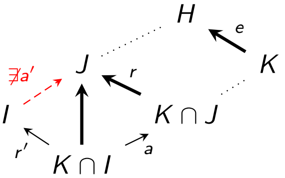
$$\textit{Every transfer } \begin{tikzcd} H \arrow[dr, dotted, no head] & \\ & J \\ K \arrow[uu, red] \arrow[ur, dotted, no head] & \end{tikzcd} \implies \textit{we also have transfers } \begin{tikzcd} H & \\ & J \arrow[ul, red] \\ K \arrow[uu] \arrow[ur] & \end{tikzcd}$$

Proof:

- If $\mathcal{O}$ is saturated then $\mathcal{M}(\mathcal{O})^c = \emptyset$.
- Every saturation failure of $\mathcal{O}$ is in $\mathcal{M}(\mathcal{O})^c = \emptyset$, thus $\mathcal{O}$ is saturated.

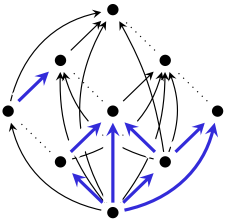
A $C_{p^2q^2}$-transfer system $\mathcal{O}$ and $\mathcal{M}(\mathcal{O})$

$\mathcal{M}(\mathcal{O})^c = \{e \in \mathcal{O} : \exists r, r' \in Res(e), a \in \mathcal{O}, \nexists a' \in \mathcal{O}$ such that



$\}.$

Can now compute non-trivial examples,



but complicated and slow, $O(r_{\mathcal{O}}^3)$ worst case with $r_{\mathcal{O}} = \#$ restrictions in $\mathcal{O}$.

A better way for a critical* case?

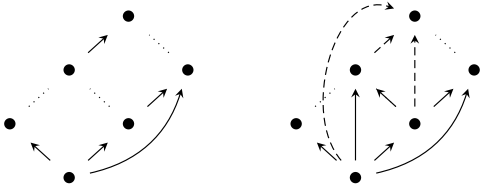
*:Additive maps of G-spectra are typically encoded by this case.

A better way for a critical* case? Disklike Transfer Systems

Definition

We say G-transfer system $\mathcal{O}$ is **disklike** when $\mathcal{O}$ is generated by transfers/relations of the form $H \to G$.

- Left, a non-disklike $C_{p^2,q}$-transfer system.
- Right, a disklike $C_{p^2,q}$-transfer system, its generators dashed.



*:Additive maps of G-spectra are typically encoded by this case.

An Observation

$$\mathcal{M}(\mathcal{O})^c = \{e \in \mathcal{O} : \exists r, r' \in Res(e), a \in \mathcal{O}, \nexists a' \in \mathcal{O} \text{ such that } \begin{tikzcd}
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If an edge $e \in \mathcal{O}$ has no non-trivial restrictions then it is in $\mathcal{M}(\mathcal{O})$.

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\end{tikzcd}\}.$$

If an edge $e \in \mathcal{O}$ has no non-trivial restrictions then it is in $\mathcal{M}(\mathcal{O})$.

Corollary

Let $(\mathcal{O}, \leq)$ be the poset of transfers of $\mathcal{O}$ ordered by restriction. Then,

$$\min_{\leq} \mathcal{O} \subseteq \mathcal{M}(\mathcal{O}).$$

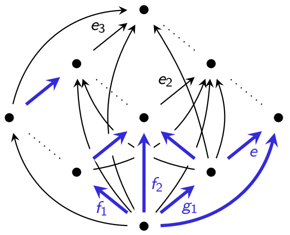
A Recursive Formula for $\mathcal{M}(\mathcal{O})$ in the Disklike Case

Theorem (DHKNSVNY)

Let $\mathcal{O}$ be a **disklike** *G-transfer system, then the maximal compatible transfer system can be written as the following set,*

$$\mathcal{M}(\mathcal{O}) = \{e \in \mathcal{O} \,|\, \forall r \prec e, r \in \mathcal{M}(\mathcal{O}) \text{ and } r \prec^S e\},$$

where $r \prec e$ means e covers r in restriction poset $(\mathcal{O}, \leq)$, and $r \prec^S e$ means $r \prec e$ satisfies the compatibility condition.



Much simpler, $O(c_{\mathcal{O}})$ worst case with $c_{\mathcal{O}} := \#$ cover relations in $(\mathcal{O}, \leq)$.

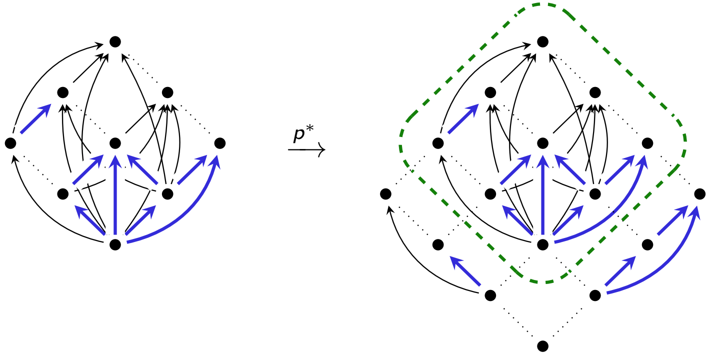
Inflation

A surjection of groups $p : G \twoheadrightarrow G/N$ where N is a normal subgroup induces a functor $p^* : Tr(G/N) \to Tr(G)$ [Rub21b].

Lemma

Inflation can be computed as $p^*(\mathcal{O}) = Res(p^{-1}(\mathcal{O}))$.

Let $p : C_{q^3r^3} \twoheadrightarrow C_{q^2r^2}$ be the projection with kernel C_{qr}, then one computes

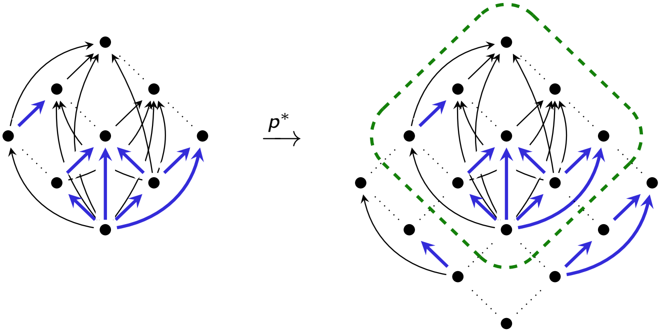


Properties Preserved by Inflation

Theorem (DHKNSVNY)

Given $p : G \twoheadrightarrow G/N$ and a pair of G/N-transfer systems $\mathcal{O}, \mathcal{O}_m$.

1. *If $\mathcal{O}$ is disklike, then so is $p^*\mathcal{O}$.*
2. *If $\mathcal{O}$ is saturated, then so is $p^*\mathcal{O}$.*
3. *If $(\mathcal{O}, \mathcal{O}_m)$ is compatible, then so is $(p^*\mathcal{O}, p^*\mathcal{O}_m)$.*
4. *If $(\mathcal{O}, \mathcal{O}_m)$ is maximally compatible, then so is $(p^*\mathcal{O}, p^*\mathcal{O}_m)$.*

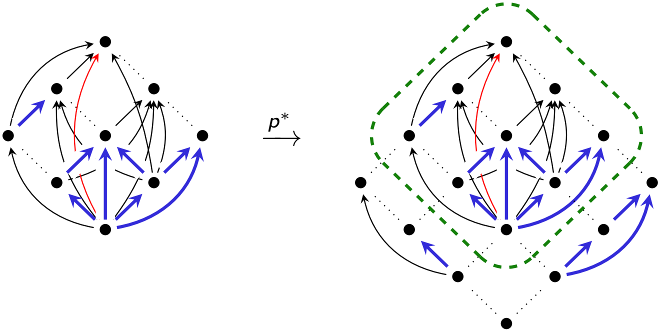


A Computational Consequence

Let $\mathcal{O}$ be a disklike G-transfer system, then there is a unique minimal subgroup with a transfer to G, $N_{\mathcal{O}} \to G$, and $N_{\mathcal{O}}$ is normal.

Corollary (DHKNSVNY)

The computation of $\mathcal{M}(\mathcal{O})$ for any disklike G-transfer system $\mathcal{O}$ can be reduced to the computation of $\mathcal{M}(\mathcal{O}')$ where $\mathcal{O}'$ is a $G/N_{\mathcal{O}}$-transfer systems with the transfer $1 \to G/N_{\mathcal{O}}$.



A Disklike Conjecture

$$\mathcal{M}(\mathcal{O}) = \{e \in \mathcal{O} \,|\, \forall r \prec e, r \in \mathcal{M}(\mathcal{O}) \text{ and } r \prec^{S} e\},$$

Conjecture

Let $\mathcal{O}$ be a disklike G-transfer system, then the maximal compatible transfer system and its complement satisfy

$$\mathcal{M}(\mathcal{O}) = \{e \in \mathcal{O} \,|\, \forall r < e, r <^{S} e\}, \text{ and}$$
$$\mathcal{M}(\mathcal{O})^c = \{e \in \mathcal{O} \,|\, \exists r < e, r <^{F} e\}.$$

Experimentally, this holds for all disklike G-transfer systems $\mathcal{O}$ such that,

1. $Order(G) \leq 15$, and $e \to G \in \mathcal{O}$,
2. $Order(G) \leq 63$, $e \to G \in \mathcal{O}$, and $Complexity(\mathcal{O}) \leq 2$, or
3. $G = \Sigma_n$ for $n \leq 6$, and $Complexity(\mathcal{O}) \leq 2$.

Clarifications and Difficulties

$$\mathcal{M}(\mathcal{O}) = \{e \in \mathcal{O} \mid \forall r \prec e, r \in \mathcal{M}(\mathcal{O}) \text{ and } r \prec^S e\},$$

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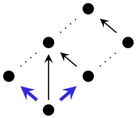
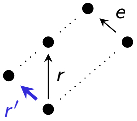
$$\mathcal{M}(\mathcal{O}) = \{e \in \mathcal{O} \mid \forall r < e, r <^S e\}, \text{ and}$$
$$\mathcal{M}(\mathcal{O})^c = \{e \in \mathcal{O} \mid \exists r < e, r <^F e\}.$$

Left, a non-disklike counter example. Right, a generic bounded lattice counter example, implying the need for group theoretic arguments if true.





Are there simpler alternate presentations?

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A Non-Disklike Counter Example

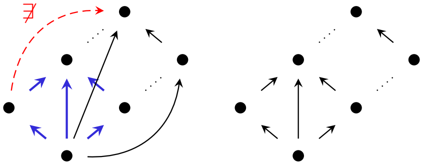
Theorem (DHKNSVNY)

Let $\mathcal{O}$ be a disklike G-transfer system, then the maximal compatible transfer system can be written as the following set,

$$\mathcal{M}(\mathcal{O}) = \{e \in \mathcal{O} \,|\, \forall r \prec e, r \in \mathcal{M}(\mathcal{O}) \text{ and } r \prec^S e\},$$

where $r \prec e$ means e covers r in restriction poset $(\mathcal{O}, \leq)$, and $r \prec^S e$ means $r \prec e$ satisfies the compatibility condition.

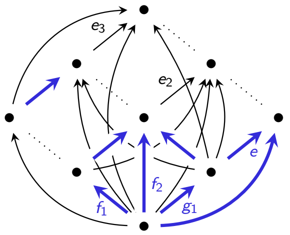
Left: a non-disklike transfer system $\mathcal{O}$ and $\mathcal{M}(O)$.
Right: the incorrect transfer system given by the theorem.



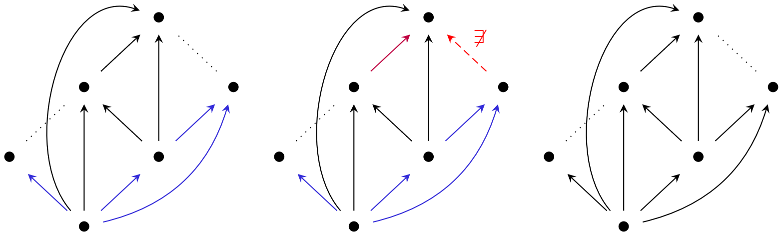
The Disklike Algorithm

Algorithm 1 Given a disklike G-transfer system $\mathcal{O}$ computes $\mathcal{M}(\mathcal{O})$.

Initialize $\mathcal{M}(\mathcal{O}) := \min \mathcal{O}$ under the $\leq$ order, and $Q := \mathcal{O} \setminus \mathcal{M}(\mathcal{O})$.
while Q is non-empty **do**
  Let m be a minimal element of Q under the $\leq$ order.
  Delete $[m]$ from Q.
  if $\forall r \in \mathcal{O}$ such that $r \prec m$, we have that $r \in \mathcal{M}(\mathcal{O})$ and $r \prec^S m$ **then**
    Add $[m]$ to $\mathcal{M}(\mathcal{O})$.
return $\mathcal{M}(\mathcal{O})$.



Saturated elements can exist in the interval $(\mathcal{M}(\mathcal{O}), \mathcal{O})$



Conjecture Example

Conjecture

Let $\mathcal{O}$ be a disklike G-transfer system, then the maximal compatible transfer system and its complement satisfy

$$\mathcal{M}(\mathcal{O}) = \{e \in \mathcal{O} \,|\, \forall r < e, r <^{S} e\}, \text{ and}$$
$$\mathcal{M}(\mathcal{O})^c = \{e \in \mathcal{O} \,|\, \exists r < e, r <^{F} e\}.$$