Compatibility with Disk like Transfer Systems II

JMM 2025

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Slides Part I

Slides Part II

Transfer Systems

Definition

Let $\mathcal{O}$ be a binary relation on $Sub(G)$ refining $\subseteq$. Then, $\mathcal{O}$ is said to be a G-transfer system if it is closed under

- conjugation,
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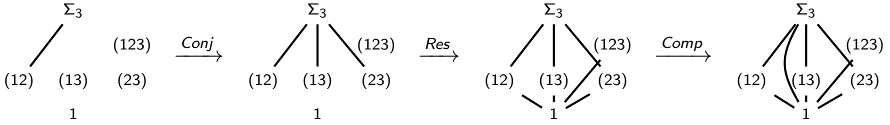
Theorem (A.2 of [Rub21])

Let R be a binary relation on $Sub(G)$ refining $\subseteq$. Let $T(R)$ denote the closure of R under

- *conjugation, then*
- *restriction, and then*
- *composition.*

Then $T(R)$ is the smallest G-transfer system containing R.

Example



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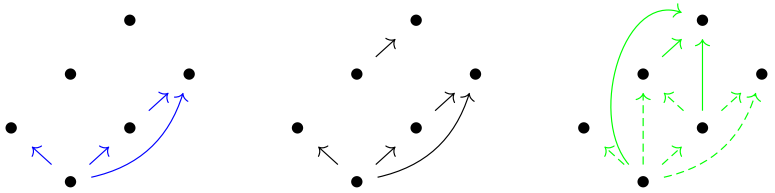
Disk like Transfer Systems

Definition

We say G-transfer system $\mathcal{O}$ is **disk like** when $\mathcal{O}$ is generated by transfers/relations of the form $H \to G$.

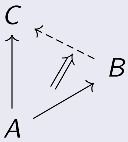
- Left, a non-disk like $C_{p^2,q}$-transfer system.
- Mid, a non-disk like $C_{p^2,q}$-transfer system.
- Right, a disk like $C_{p^2,q}$-transfer system, its generators in solid green.



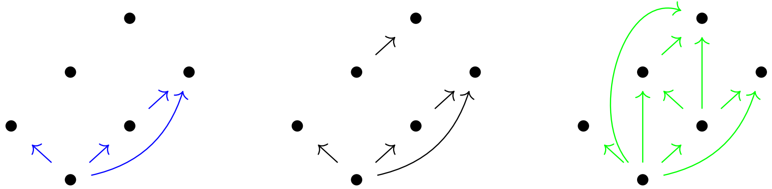
Saturated Transfer Systems

Definition

A transfer system $\mathcal{O}$ is **saturated** if it satisfies the 2 out of 3 property.



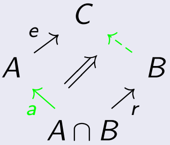
- Left, a saturated $C_{p^2,q}$-transfer system.
- Mid, a saturated $C_{p^2,q}$-transfer system.
- Right, a non-saturated $C_{p^2,q}$-transfer system.



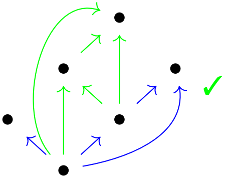
Compatible Transfer Systems

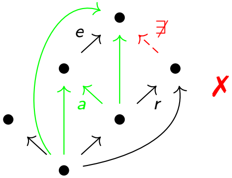
Definition ([Cha24, Definition 4.6])

Let $\mathcal{O}_a$ and $\mathcal{O}_m$ be a pair of G-transfer systems such that $\mathcal{O}_m \subseteq \mathcal{O}_a$. We say $(\mathcal{O}_a, \mathcal{O}_m)$ are **compatible** if we can complete all squares of the form



with $e, r \in \mathcal{O}_m$ and $a \in \mathcal{O}_a$.





Maximal Compatible Transfer Systems

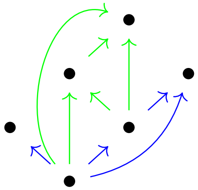
Proposition ([BH22])

If $(\mathcal{O}_a, \mathcal{O}_m)$ and $(\mathcal{O}_a, \mathcal{O}'_m)$ are both compatible, then $(\mathcal{O}_a, \mathcal{O}_m \vee \mathcal{O}'_m)$ is compatible.

Corollary

For a fixed transfer system $\mathcal{O}_a$,

- *there exists a maximal compatible transfer system $\mathcal{O}_m$, and*
- *all other compatible transfers systems are sub-transfer systems of $\mathcal{O}_m$.*



Why?

Work including [BH15, GW18, BBR21, BP21, Rub21, BMO24, Cha24], provides the correspondences

N_∞-operads	Transfer Systems
Additive Transfers	Disk Like Transfer Systems
Multiplicative Norms	Saturated Transfer Systems
Bi-incomplete Transfers and Norms	Compatible Transfer Systems

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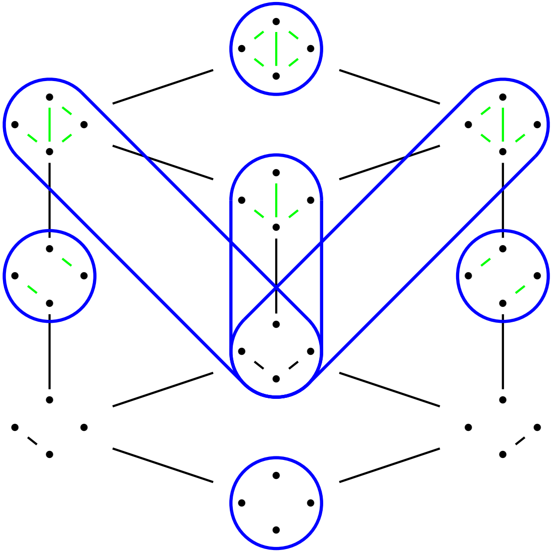
Thus identifying the maximal compatible transfer system identifies all bi-incomplete/compatible multiplicative norms for a fixed additive transfer.

Maximal Compatible Pairs of Disk like Transfers of $C_{p,q}$



How?

In Part 1 with David, we saw that

Proposition (DHKNSVNY)

The maximal compatible transfer $\mathcal{O}_m$ of $\mathcal{O}_a$ is always saturated.

Proposition (DHKNSVNY)

A transfer system $\mathcal{O}_a$ is self compatible, if, and only if, it is saturated.

Is $\mathcal{O}_m$ the 'maximal saturated sub-transfer system' of $\mathcal{O}_a$?

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Is $\mathcal{O}_m$ the 'maximal saturated sub-transfer system' of $\mathcal{O}_a$?

No!

1. There can exist multiple incomparable saturated transfer systems smaller than $\mathcal{O}_a$.
2. Saturated elements can exist in the open interval $(\mathcal{O}_m, \mathcal{O}_a)$.

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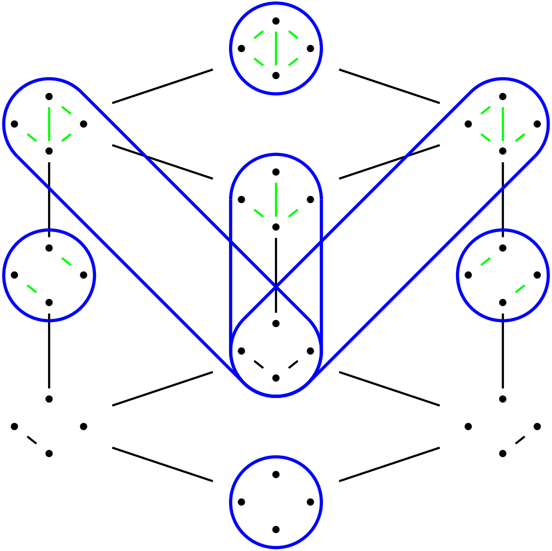
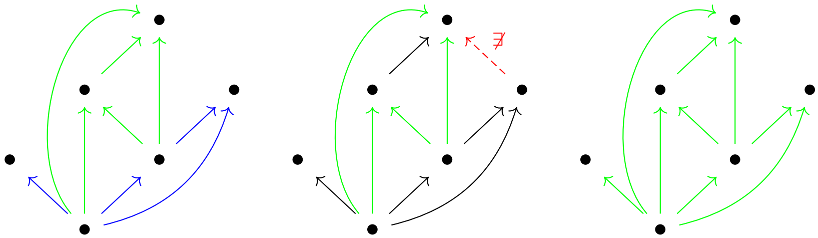


Saturated elements can exist in the open interval $(\mathcal{O}_m, \mathcal{O}_a)$



Computing $\mathcal{O}_m$

Worst case need to check $\begin{array}{ccc} & C & \\ A \overset{e}{\nearrow} & \Nearrow & \nwarrow B \\ \underset{a}{\nwarrow} & A \cap B & \underset{r}{\nearrow} \end{array}$ for all $e, r \in \mathcal{O}_m$ and $a \in \mathcal{O}_a$.

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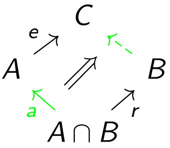
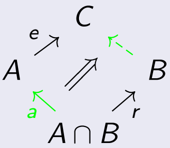


for all $e, r \in \mathcal{O}_m$ and $a \in \mathcal{O}_a$.

Lemma (DHKNSVNY)

If $\mathcal{O}_a = Comp(Res(Conj(B_a)))$ and $\mathcal{O}_m = Comp(Res(Conj(B_m)))$. Then, $(\mathcal{O}_a, \mathcal{O}_m)$ are compatible, if, and only if, $\mathcal{O}_m \subseteq \mathcal{O}_a$ and



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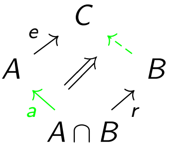

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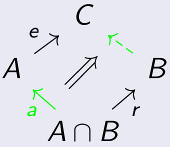


for all $e, f \in Res(Conj(B_m))$ and $a \in Res(Conj(B_a))$.

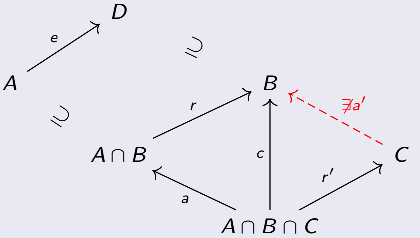
Also: It is possible to not conjugate one of the sets of generators!

Computing $\mathcal{O}_m$ Via Its Complement

Proposition (DHKNSVNY)

The complement of the maximal compatible transfer system of $\mathcal{O}_a$ satisfies

$$\mathcal{O}_m^c := \mathcal{O}_a \setminus \mathcal{O}_m = \{e \in \mathcal{O}_a : \exists r, r' \in Res(e), a \in \mathcal{O}_a, \nexists a' \in \mathcal{O}_a \text{ such that}$$



$$\}.$$

Idea:

- Delete the composite and top left factors of saturation failures of $\mathcal{O}_a$.
- Every occurrence of the pattern above deletes e, r and c.

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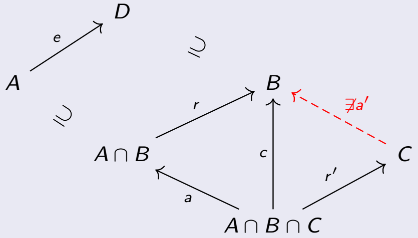

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Proof sketch:

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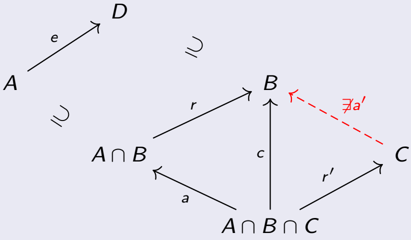

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Proof sketch:

- $e \in \mathcal{O}_m$ if, and only if, $(\mathcal{O}_a, T(e))$ is compatible.

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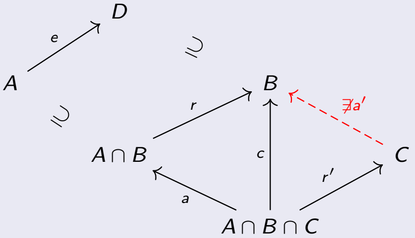

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Proof sketch:

- $e \in \mathcal{O}_m$ if, and only if, $(\mathcal{O}_a, T(e))$ is compatible.
- Then check the compatibility of $(\mathcal{O}_a, T(e))$ with prior lemma.

Case: $\mathcal{O}_m^c = \emptyset$

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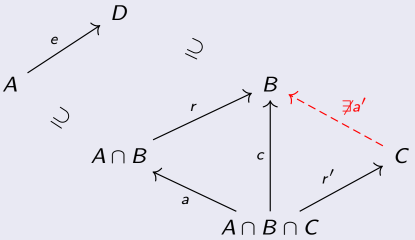

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Proposition (DHKNSVNY)

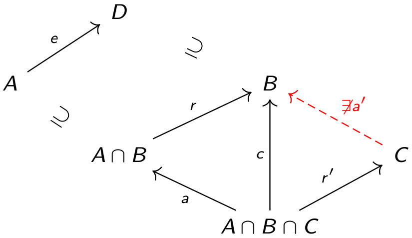
A transfer system $\mathcal{O}_a$ is self compatible, if, and only if, it is saturated.

Proof:

- If $\mathcal{O}_a$ is saturated then $\mathcal{O}_m^c = \emptyset$.
- Every saturation failure of $\mathcal{O}_a$ is in $\mathcal{O}_m^c = \emptyset$, thus $\mathcal{O}_a$ is saturated.

Example: Computing $\mathcal{O}_m$ in C_{p^2,q^2} using the complement

$\mathcal{O}_m^c := \mathcal{O}_a \setminus \mathcal{O}_m = \{e \in \mathcal{O}_a : \exists r, r' \in Res(e), a \in \mathcal{O}_a, \nexists a' \in \mathcal{O}_a$ such that



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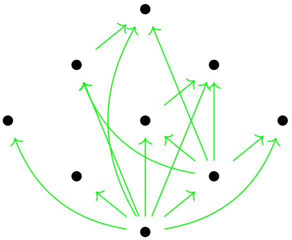

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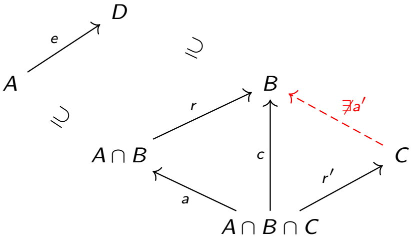

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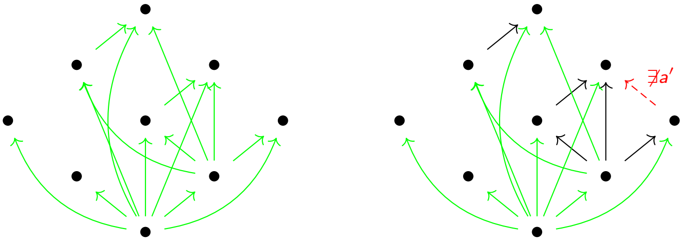

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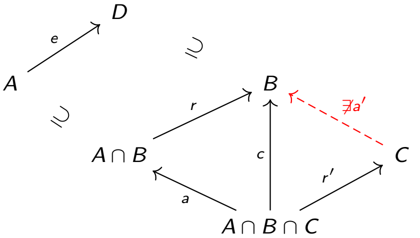

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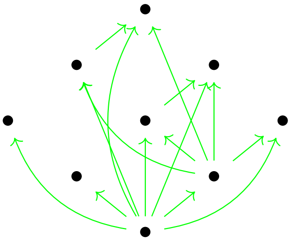

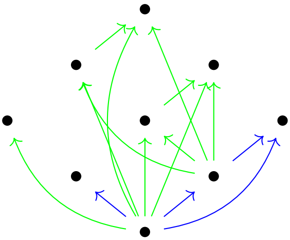

A Key Consequence

Proposition (DHKNSVNY)

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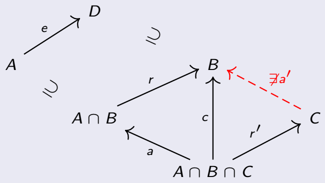

$\}.$

Theorem (DHKNSVY)

Let $\phi : G \to G'$ *be a group hom. and* $\mathcal{O}, \mathcal{O}_a, \mathcal{O}_m$ G'*-transfer systems.*

1. *If* $\mathcal{O}$ *is disklike [resp. saturated], so is* $\phi^*\mathcal{O}$*.*
2. *If* $(\mathcal{O}_a, \mathcal{O}_m)$ *is compatible and* $\mathcal{O}_m$ *is saturated, then* $(\phi^*\mathcal{O}_a, \phi^*\mathcal{O}_m)$ *is compatible.*
3. *If* $(\mathcal{O}_a, \mathcal{O}_m)$ *is maximally compatible, then so is* $(\phi^*\mathcal{O}_a, \phi^*\mathcal{O}_m)$*.*

Some Questions and Future Directions

- How can we use the disk like assumption to aid in computing $\mathcal{O}_m$?
- Can we use subset bounds on $\mathcal{O}_m$ for faster computation?
- Can we compute all maximal compatible pairs for all disk like transfer systems of a fixed group G in a relatively efficient manner?
  - i.e. maybe it is 'hard' to compute $\mathcal{O}_m$ for arbitrary $\mathcal{O}_a$,
  - but we can induct to all $(\mathcal{O}_a, \mathcal{O}_m)$ from computing $(T(H \to G), \mathcal{O}_m)$?

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