Combinatorial Techniques in Operadic Homotopy Theory

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Slides

Thesis

Overview

- Background
  - What are Operads?
  - What are Operadic Structures?
  - What is Operadic Homotopy Theory?
    - With an algebraic leaning...
- Delve into the two papers comprising this Thesis
  - "Koszul Operads Governing Props and Wheeled Props", and
  - "Diagonals of the Permutahedra", joint with Bérénice Delcroix-Oger, Guillaume Laplante-Anfossi and Vincent Pilaud.

Operads

Definition (Informal)

An **operad** P is a sequence of sets $(P(n))_{n\in\mathbb{N}}$ with maps

$$P(n) \boxtimes P(k) \xrightarrow{\circ_i} P(n+k-1) \quad \text{for } 1 \leq i \leq n,$$

which satisfy tree like associativity axioms.

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Example

Let $(\mathcal{E}, \boxtimes)$ be a symmetric monoidal category, and let X be an object of $\mathcal{E}$. The **endomorphism operad** End_X is defined $End_X(n) := Hom_{\mathcal{E}}(X^{\boxtimes n}, X)$.

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- If $(\mathcal{E}, \boxtimes) := (Set, \times)$ and X is a set, then

$$End_X(n) \text{ is the set of maps of sets } X^{\times n} \to X.$$

- If $(\mathcal{E}, \boxtimes) := (Vect_{\mathbb{K}}, \otimes)$ and X is a Vector Space, then

$$End_X(n) \text{ is the set of linear transformations } X^{\otimes n} \to X.$$

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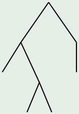
which satisfy tree like associativity axioms.

Example

The **non-symmetric associative operad** As in $(Set, \times)$,

$As := F($  $)/\langle$  $\overset{r_1}{=}$  $\rangle.$

$As(n) =$ The set of n-ary binary planar trees modulo r_1,

with $\circ_i$ given by grafting  $\circ_2$  $=$  .

Algebras Over Operads

Definition

Given operads P, Q in $(\mathcal{E}, \boxtimes)$, a **morphism of operads** $\theta : P \to Q$ is a sequence of maps $\theta := (\theta_n : P(n) \to Q(n))_{n \in \mathbb{N}}$ such that for all $\alpha, \beta \in P$

$$\theta(\alpha \circ_i \beta) = \theta(\alpha) \circ_i \theta(\beta).$$

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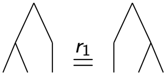
Idea,

- $\wedge \in As(2)$ is concretely realised as $A(\wedge) : X \times X \to X$, and
- the relation  forces $A(\wedge)$ to be associative.

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A P-**algebra** is a morphism of operads $A : P \to End_X$.

Example

Let $\mathbb{K}$ be a field, and define $\mathbb{K}As$ to be the $\mathbb{K}$-linear span of As, i.e.

$$\mathbb{K}As(n) := \mathbb{K}\langle As(n)\rangle.$$

In $(Vect_{\mathbb{K}}, \otimes)$ a $\mathbb{K}As$-algebra $A : \mathbb{K}As \to End_X$ is an associative algebra.

Operadic Structures

Definition (Informal)

Operadic structures model the composition of different types of functions through different types of graphs. E.g.

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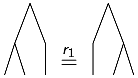
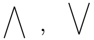
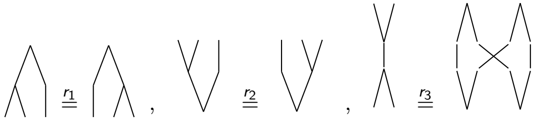
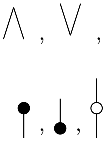
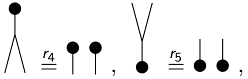
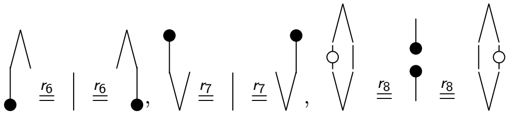
- **Props** model the composition of functions with many inputs and many outputs via (possibly disconnected) directed acyclic graphs.

They each admit generalised definitions of morphisms and algebras.

Operadic Structures: *Operads* $\subsetneq$ *Properads* $\subsetneq$ *Props*
Trees $\subsetneq$ *Conn.* $\subsetneq$ *Disconn.*

An Example Element of Operadic Family		Its Algebras in Vect
Generators	Relations	
		As. Algebras
		Bialgebras
	r_1, r_2, r_3 and,  	Hopf Algebras

Can We Also Classify Structures with Non-Strict Relations?

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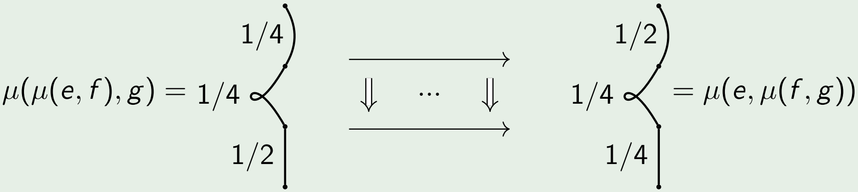
Example (Path Spaces)

If X is a topological space, let $P(X) := Hom_{Top}([0,1], X)$ be the set of continuous functions from the interval into X equipped with the compact-open topology, and let $\mu : P(x) \boxtimes P(x) \to P(x)$ be defined by

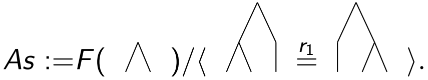
$$\mu(p_1, p_2) := \begin{cases} p_1(2t), & 0 \leq t \leq 1/2 \\ p_2(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

Then μ is only associative up to homotopy,



Operadic Homotopy Theory

Idea: Take an operadic structure, such as



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This results in an infinite tower of higher homotopies.

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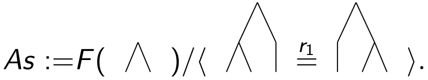

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Two common problems in operadic homotopy theory,

1. Parse this tower of higher homotopies.
2. Extend known constructions to homotopy weakened generalisations.

The Algebraic Setting

Definition (See for instance [DCV13])

An **algebraic operad** P is defined to be an operad in $(dgVect_{\mathbb{K}}, \otimes)$. Given two algebraic operads M, P we say that M is a **model** for P if

- there exists a morphism of operads $f : M \to P$, and
- f is both a quasi-isomorphism, and an epimorphism.

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Proposition ([DCV13])

When a minimal model of an algebraic operad P exists, it is unique up to isomorphism.

A minimal model is the best/smallest possible cofibrant replacement in the model category of algebraic operads [Hin97].

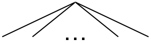
The A_∞-Operad

Theorem ([Sta63], for a modern presentation see [LV12])

There exists a quadratic model for $\mathbb{K}As$, the A_∞-operad.

$$A_\infty := (F((\mu_n)_{n\geq 1}), d).$$

where

- μ_n is a n-ary operation also denoted $\mu_n =$ ,
- μ_1 encodes a differential,
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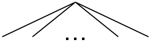
- μ_n is a n-ary operation also denoted $\mu_n =$ ,
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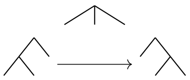
Thus A_∞-algebras can thus be seen as homotopy associative algebras.

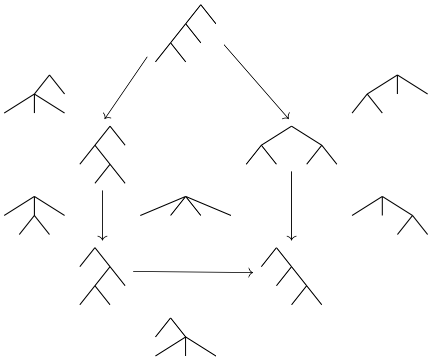
The A_∞-Operad

Definition

A **convex polytope** is the convex hull of a collection of points in $\mathbb{R}^n$.

The **associahedra** $\mathcal{K} = (\mathcal{K}_n)_{n \in \mathbb{N}}$ is a cell complex which can be realised as convex polytope whose faces are in bijection with planar trees.





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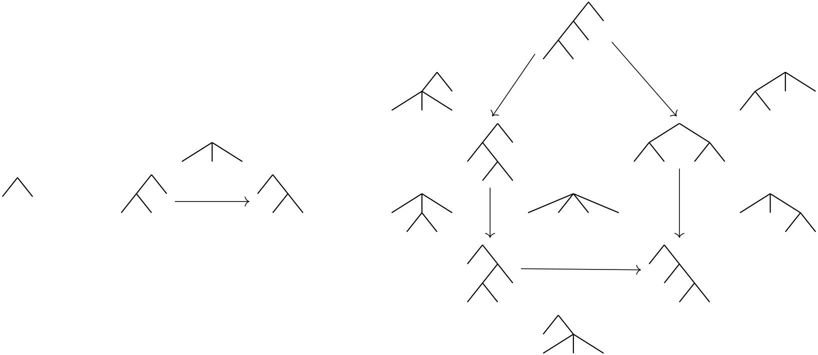
The associahedra $\mathcal{K}$ encodes the derivations of the A_∞-operad.



Koszul Operads, see for instance [LV12]

Definition

An algebraic operad P is **Koszul**, if, and only if, it has a quadratic model.

Many other characterisations and consequences!

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- $A_\infty = \Omega((\mathbb{K}As)^{¡})$, as the associative operad is Koszul self-dual, an A_∞-algebra is a codifferential on a cofree coassociative coalgebra.

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- Any homotopy retract of a P-algebra is a P_∞-algebra. For example the homology of a P-algebra has an explicit P_∞-structure.
- It provides $Ho(P\text{-}alg) \cong Ho(\infty\text{-}P_\infty\text{-}alg)$, and explicit maps to resolve P-algebras into P_∞-algebras, and rectify P_∞-algebras into P-algebras.

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- The homology of an associative algebra has Massey products.
- Recovers classical bar construction, and classical rectification results.

Koszul Operads Governing Operadic

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There exist (groupoid) coloured operads governing all operadic structures.

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Theorem ([BM23], [KW23])

The groupoid coloured operads governing connected operadic structures

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Theorem ([Sto24])

The groupoid coloured operads governing props and wheeled/traced props

1. *are Koszul,*
2. *are not Koszul self-dual, and*
3. *do not have quadratic models governed by polytopes.*

How?

- Define props.
- Describe the coloured operad $\mathbb{P}$ whose algebras are props.
- Reinterpret $\mathbb{P}$ as a groupoid coloured operad.
- Show $\mathbb{P}$ admits a quadratic presentation $\mathbb{P} \cong F(E)/\langle R \rangle$.
  - i.e. every term in R contains two generators of E, for example like

$As := F($  $)/\langle$  $\overset{r_1}{=}$  $\rangle.$

- Show $\mathbb{P}$ is Koszul through the following general result.

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Let P be a groupoid coloured operad such that the associated coloured shuffle operad $(P^f)_*$ admits a quadratic Groebner basis, then P is Koszul.

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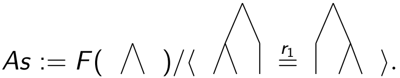

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Will give a simple example of a groupoid coloured operad, and these maps at end if time/interest!

Props

Let $\mathfrak{C}$ be a set of colours, and denote a sequence of colours $\underline{c} = (c_1, ..., c_n)$

Definition (Introduced in [ML65], in one coloured *dgVect* case)

A $\mathfrak{C}$-coloured **prop** P is a strict symmetric monoidal category whose objects are generated by the free monoid $F(\mathfrak{C})$.

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- P is a symmetric bimodule, i.e. if $\alpha \in P\binom{\underline{d}}{\underline{c}}$ then $\alpha \cdot \binom{\sigma}{\tau} \in P\binom{\sigma\underline{d}}{\underline{c}\tau}$.
- P has a vertical composition $P\binom{\underline{c}}{\underline{b}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\circ_V} P\binom{\underline{c}}{\underline{a}}$.
- P has a horizontal composition $P\binom{\underline{d}}{\underline{c}} \otimes P\binom{\underline{b}}{\underline{a}} \xrightarrow{\circ_h} P\binom{\underline{d},\underline{b}}{\underline{c},\underline{a}}$.

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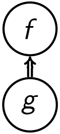
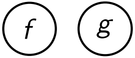
Can graphically visualise $\circ_V(f, g)$ as 

, and $\circ_h(f, g)$ as 

The Operad Governing Props

Proposition (Specialisation of Lemma 14.2 [YJ15])

There exists an operad $\mathbb{P}$ in $\mathsf{Vect}_{\mathbb{K}}$ whose algebras are $\mathfrak{C}$-props in $\mathsf{Vect}_{\mathbb{K}}$.

The Operad Governing Props

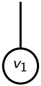
Generalised Graphs $\approx$ Vertices + Flags/Half-Edges

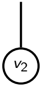
Proposition (Specialisation of Lemma 14.2 [YJ15])

There exists an operad $\mathbb{P}$ in $Vect_{\mathbb{K}}$ whose algebras are $\mathfrak{C}$-props in $Vect_{\mathbb{K}}$. If $Gr^{\uparrow}$ is the set of strict isomorphism classes of directed, vertex labelled generalised graphs, and with no directed cycles, then $\mathbb{P}$ has operations

$$\mathbb{P}\left(\substack{\binom{\underline{d}}{\underline{c}} \\ \binom{\underline{d}_1}{\underline{c}_1}, \dots, \binom{\underline{d}_k}{\underline{c}_k}}\right) := \mathbb{K}\langle\{\gamma \in Gr^{\uparrow} : \gamma \text{ profile } \binom{\underline{d}}{\underline{c}},\ v_i \text{ profile } \binom{\underline{d}_i}{\underline{c}_i}\}\rangle$$

$\in \mathbb{P}\left(\substack{\binom{0}{0} \\ \binom{0}{2}, \binom{2}{0}}\right)$,





$\in \mathbb{P}\left(\substack{\binom{2}{0} \\ \binom{1}{0}, \binom{1}{0}}\right)$,

The Operad Governing Props

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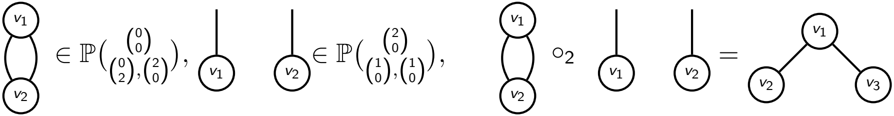
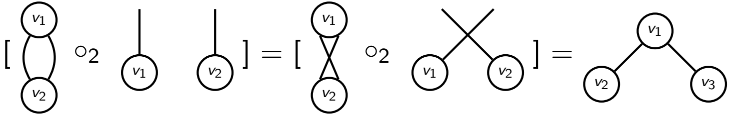
and a partial composition $\circ_i$ given by graph substitution.



Towards a Quadratic Presentation

- We work with non-unital version of $\mathbb{P}$, i.e. assume all graphs have at least two vertices.
- We re-interpret $\mathbb{P}$ as being a groupoid coloured operad, for instance



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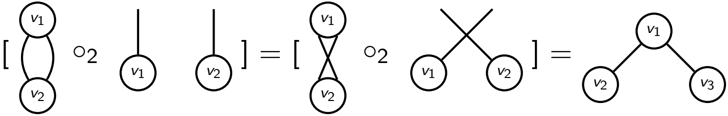


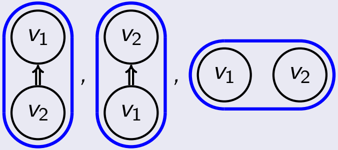
Without these assumptions the resulting presentation would have quadratic unary relations. See [DV21] for case of operad governing operads.

*Thus the resulting notion of a $\mathbb{P}_\infty$ algebra doesn't relax unit or equivariance axioms up to homotopy.

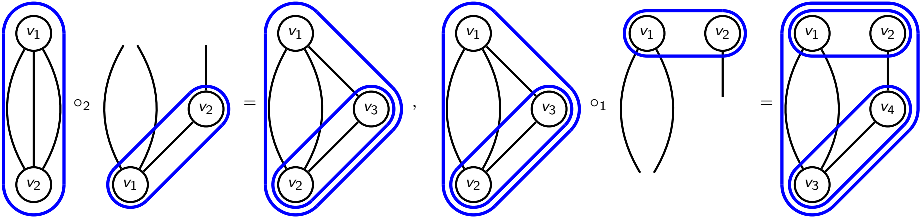
The Free Full Nesting Operad

Proposition

Let E be $\mathbb{K}$-linear span of graphs with two vertices, partitioned as follows.

$E = \mathbb{K}\langle\{$  $\}\rangle.$

The free operad $F(E)$ is equivalent to 'fully nested graphs' with $\circ_i$ given by nested graph substitution.

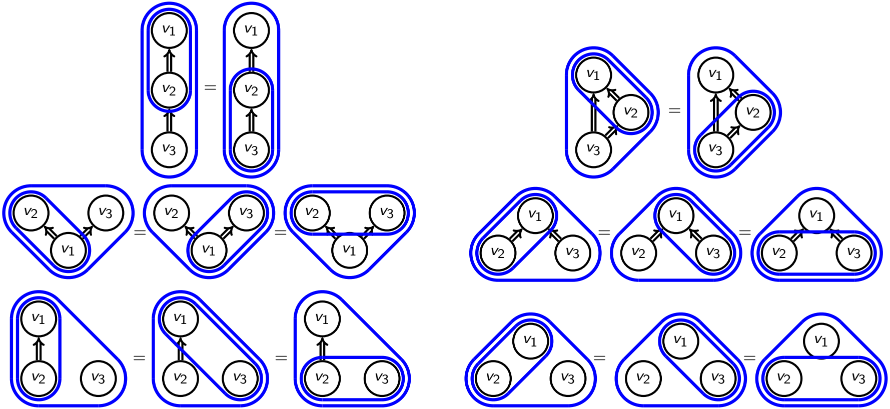


A Quadratic Presentation of the Operad Governing Props

Proposition

The groupoid coloured operad $\mathbb{P}$ admits a quadratic presentation $\mathbb{P} \cong F(E)/\langle R \rangle$.

The quadratic relations R are,



Proving Koszul via Rewriting Techniques

Theorem ([Sto24])

Let P be a groupoid coloured operad such that the associated coloured shuffle operad $(P^f)_*$ admits a quadratic Groebner basis, then P is Koszul.

Will outline why this is true at end of talk if time.

Idea: A straightforward combinatorial condition for being Koszul.

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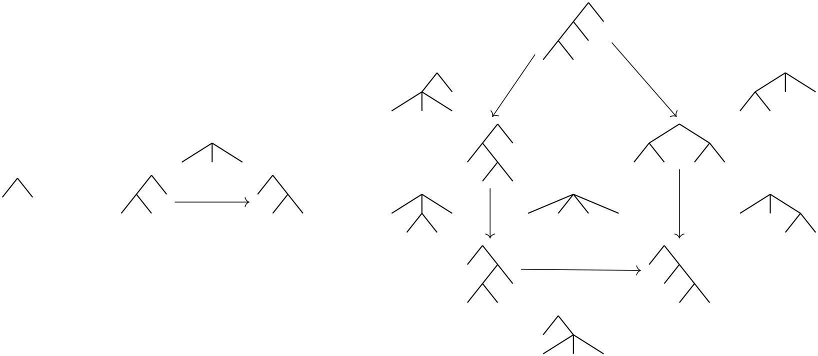
Idea: A straightforward combinatorial condition for being Koszul.

An operad with a quadratic presentation $F(E)/\langle R\rangle$ is Koszul if we can direct the relations R into a rewriting system on $F(E)$ which is

- **terminating**, i.e. no infinite chain of rewrites, and
- **confluent**, i.e. divergent rewrites eventually converge.

Proving Koszul via Rewriting Techniques

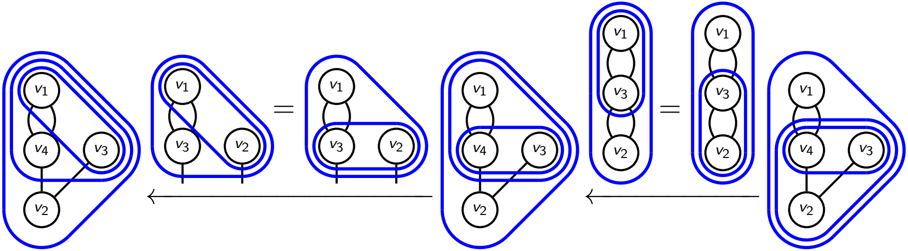
Directing $As = F($  $)/\langle$  $\xrightarrow{r_1}$  $\rangle$, the edges of $\mathcal{K}$ correspond to the resulting rewrite system, which is confluent and terminating.



Proving Koszul via Rewriting Techniques

The operad $\mathbb{P}$ admits such a confluent terminating rewrite system.

- Every graph has a unique minimal tree monomial forming it.
- Every non-minimal tree monomial can be rewritten to the minimal tree via the directed relations of $\mathbb{P}$.



Polytopes and the Koszul Machine Parse Higher Structure

Theorem ([BM23], [KW23])

The groupoid coloured operads governing connected operadic structures

1. *are Koszul,*
2. *are Koszul self-dual, and*
3. *have quadratic models governed by polytopes.*

Theorem ([Sto24])

The groupoid coloured operads governing props and wheeled/traced props

1. *are Koszul,*
2. *are not Koszul self-dual, and*
3. *do not have quadratic models governed by polytopes.*

However the additional geometry of polytopes provides further data...

Tensor Products of Associative Algebras

Proposition

Let A and B be two associative algebras, then $A \otimes B$ is also an associative algebra, with product

$$\mu(a \otimes a', b \otimes b') := \mu_A(a, a') \otimes \mu_B(b, b')$$

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Let A and B be two A_∞-algebras, then $A \otimes B$ is also an A_∞-algebra.

But how do we explicate all the higher products $(\mu_n)_{n \in \mathbb{N}}$?

Joint with:

Bérénice Delcroix-Oger, Guillaume Laplante-Anfossi and Vincent Pilaud.

How do we explicate the $(\mu_n)_{n\in\mathbb{N}}$ in $A \otimes B$?

Theorem ([SU04],[MS06],[MTTV21],[LA22],[LAM23],...)

Formulae for coherent cellular diagonals of

- *the associahedra, defines a tensor product of A_∞-algebras,*
- *the multiplihedra, defines a tensor product of A_∞-morphisms, and*
- *the operahedra, defines a tensor product of homotopy operads.*

How do we explicate the $(\mu_n)_{n\in\mathbb{N}}$ in $A \otimes B$?

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These are canonical projections of the permutahedra.

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Theorem ([DOLAPS23])

There are exactly two geometric universal tensor products of:

- *A_∞-algebras,*
- *A_∞-morphisms, and*
- *(non-symmetric non-unital) homotopy operads.*

In each case, both tensor products are ∞-isotopic.

Cellular Diagonals

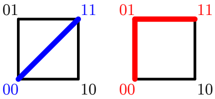
Definition

A **convex polytope** is the convex hull of a collection of points in $\mathbb{R}^n$.

Definition

Let P be a convex polytope. The **thin diagonal** $\triangle : P \to P \times P$ is defined by $\triangle(x) := (x, x)$. A **cellular diagonal** is a cellular approximation of $\triangle$, i.e.

- its image is a union of cells of $P \times P$, and
- it approximates $\triangle$ up to homotopy, in particular agreeing on vertices.



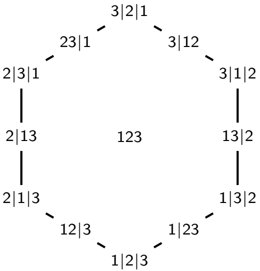
The thin and a cellular diagonal of the interval $[0, 1]$.

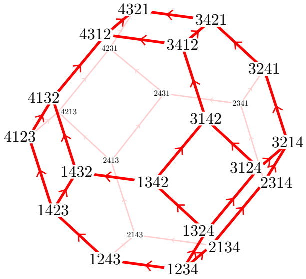
# of Permutahedra

Definition

The permutahedra $\mathcal{P}_n$ is the convex hull of the points

$$(\sigma(1), ..., \sigma(n)) \in \mathbb{R}^n, \sigma \in \mathbb{S}_n$$





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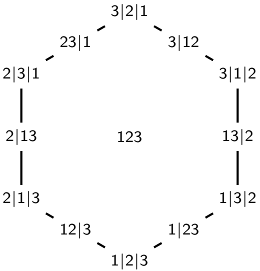

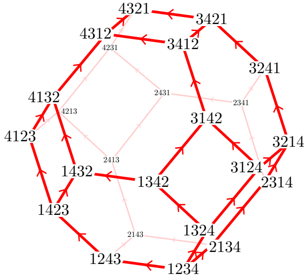

Each face is in bijection with ordered $[n]$-partitions, and there exists an isomorphism Θ which decomposes each face $A_1| \ldots |A_k$ of the permutahedron $\mathcal{P}_{|A_1|+\cdots+|A_k|-1}$ as a product $\mathcal{P}_{|A_1|-1} \times \cdots \times \mathcal{P}_{|A_k|-1}$.

Coherent Cellular Diagonals

There exists an isomorphism Θ which decomposes each face $A_1|\ldots|A_k$ of the permutahedron $\mathcal{P}_{|A_1|+\cdots+|A_k|-1}$ as a product $\mathcal{P}_{|A_1|-1} \times \cdots \times \mathcal{P}_{|A_k|-1}$.

Definition

A cellular diagonal of the permutahedra $\triangle$ is **coherent** if for every face $A_1|\ldots|A_k$ of the permutahedron $\mathcal{P}_{|A_1|+\cdots+|A_k|-1}$, the map Θ induces a topological cellular isomorphism

$$\triangle(A_1) \times \ldots \times \triangle(A_k) \cong \triangle(A_1|\ldots|A_k) \ .$$

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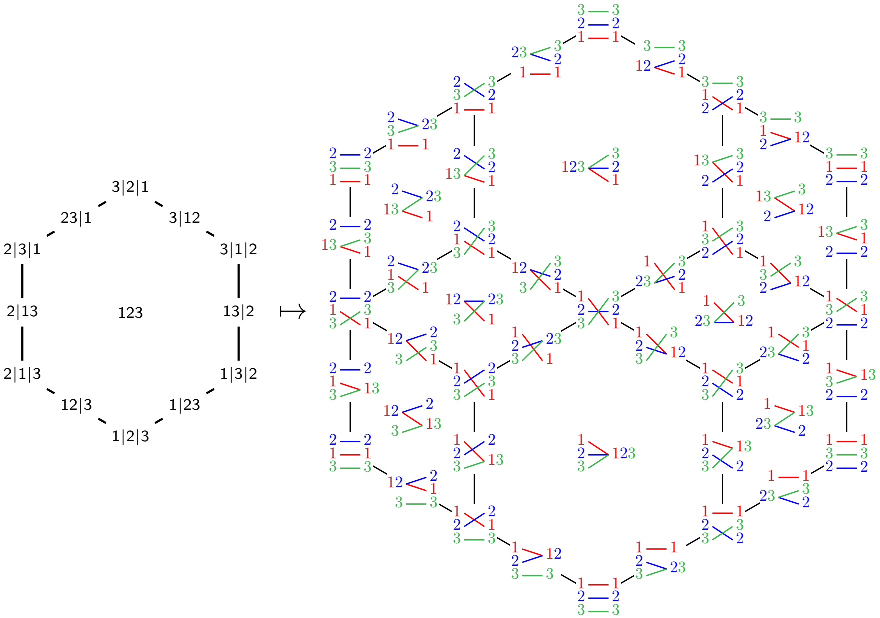
Theorem ([DOLAPS23])

There are exactly four coherent cellular diagonals of the permutahedra

1. *the LA diagonal of [LA22],*
2. *the SU diagonal of [SU04],*

and their $-^{op}$ orders. Moreover, their face posets are isomorphic lattices.

The Image of the LA and SU Diagonals of $\mathcal{P}_3$

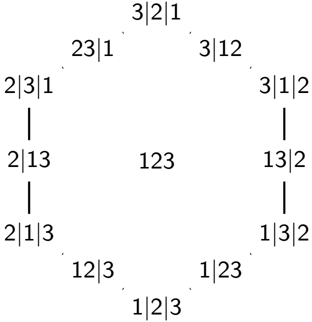


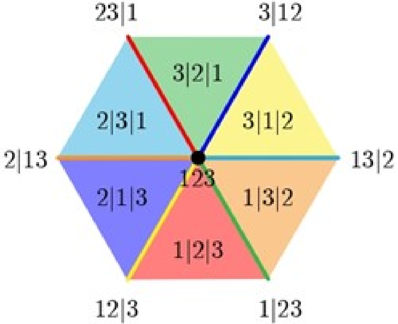
Computing the Diagonals: A Useful Duality

Definition

A **hyperplane** arrangement is a finite set $\mathcal{H}$ of affine hyperplanes in $\mathbb{R}^d$.

The **normal cone** of $\mathcal{P}_n$ is the **braid arrangement** $\mathcal{B}_n$.

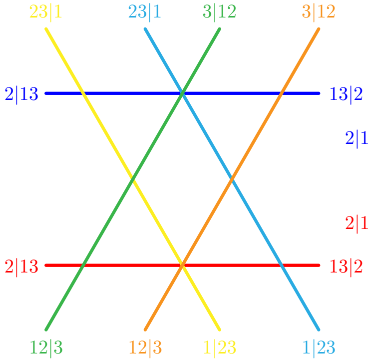


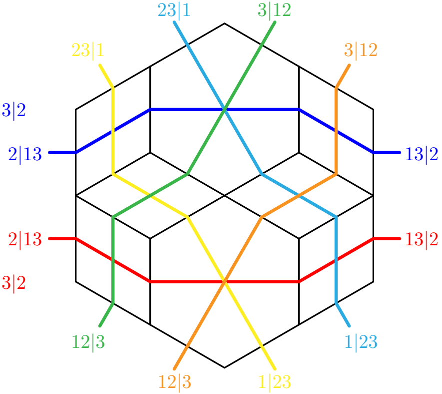


The hyperplanes, regions, and intersections of $\mathcal{B}_3$ are labelled via their duality with $\mathcal{P}_3$.

Computing the Diagonals: A Useful Duality

Idea: This duality extends to the diagonal.





A cellular diagonal of $\mathcal{P}_n$ is the dual of a hyperplane arrangement $\mathcal{B}_n^2$ consisting of two <u>generically</u> translated copies of $\mathcal{B}_n$.

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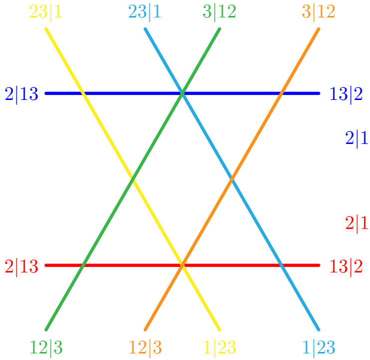

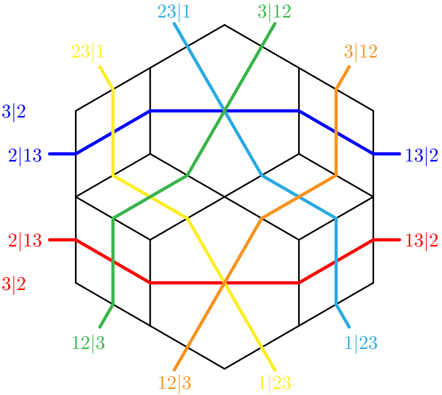

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The *LA* and *SU* diagonals are particular translations [LA22], [DOLAPS23].

An Explicit Formula

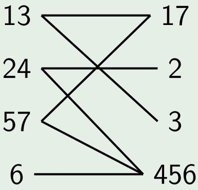
Definition

A n-partition tree is a pair (σ, τ) of set partitions of $[n]$ whose intersection graph is a bipartite tree.

Example

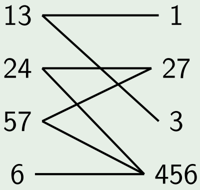
An example and counter example,

$13|24|57|6 \times 17|2|3|456$



✓

$13|24|57|6 \times 1|27|3|456$



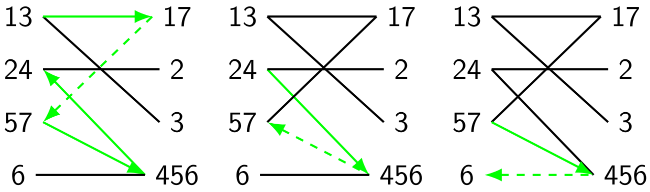
✗

Theorem ([DOLAPS23])

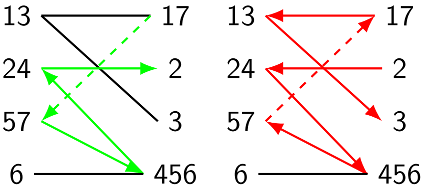
Let (σ, τ) be a pair of ordered partitions of $[n]$ forming an n-partition tree. If for all pairs of adjacent blocks, the directed path between them traverses

1. *the* **maximal** *path element* **right to left**, *then $(\sigma, \tau) \in \triangle^{SU}$.*

σ is Good:



τ is Bad:



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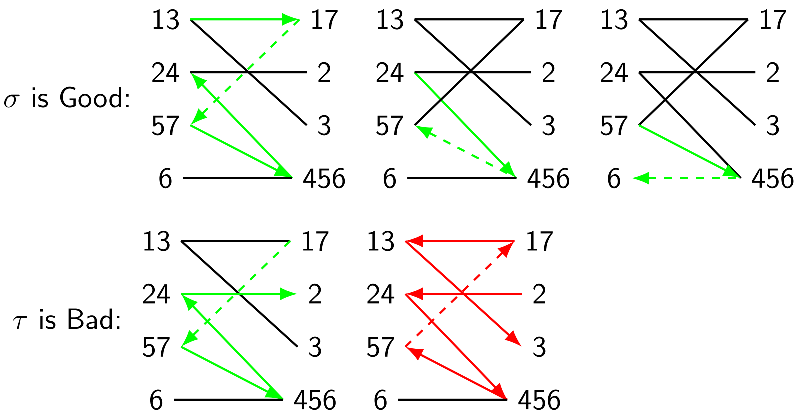
2. *the* **minimal** *path element* **left to right**, *then $(\sigma, \tau) \in \triangle^{LA}$.*



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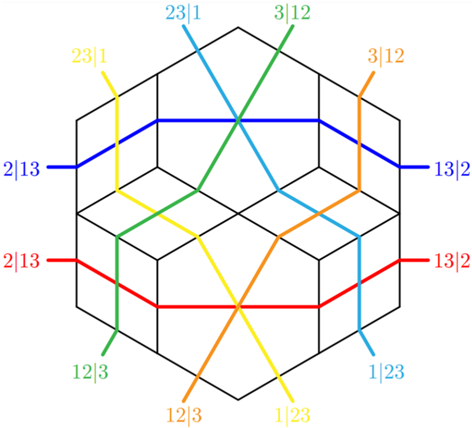

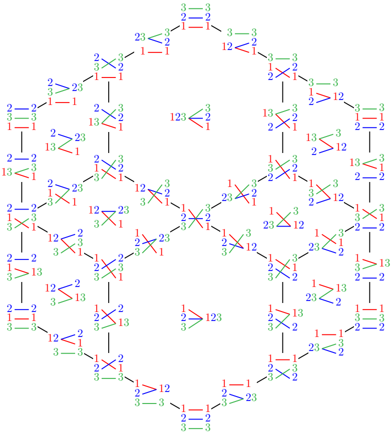

Connecting Previously Disparate Formulae

Theorem ([DOLAPS23])

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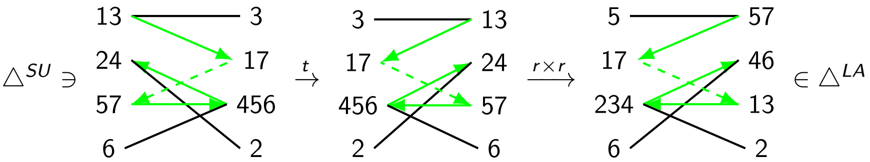
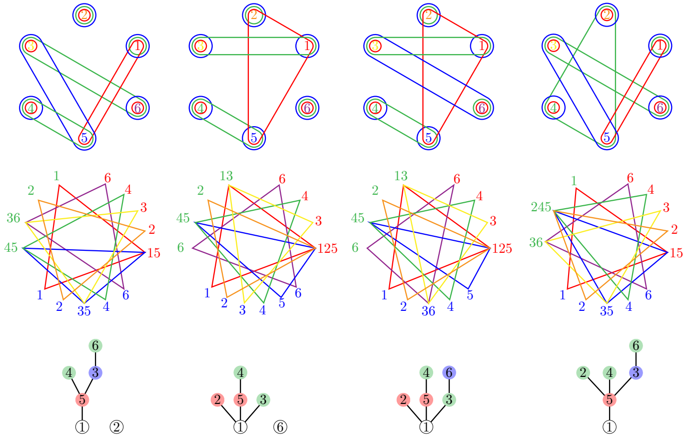


- The path formulae.
- The geometric formulae of [LA22].
- The shift formulae of [SU04].
- The cubical formulae of [SU04].
- The matrix formulae of [SU04].

$\mathcal{B}_n^\ell$: ℓ-generically translated copies of the braid arrangement

Our paper also treats the ℓ-ary case using [Zas75], obtaining complete enumerations, and combinatorial characterisations via rainbow trees/forests, and generalised Prüfer codes.



A Thought For The Future

Theorem ([BM23], [KW23])

The groupoid coloured operads governing connected operadic structures

1. *are Koszul,*
2. *are Koszul self-dual, and*
3. *have quadratic models governed by* **polytopes***.*

Can we systemically construct and study the diagonals, and resulting tensor products of all homotopy operadic structures?

Mentioned Sources I

[BM23] Michael Batanin and Martin Markl. Koszul duality for operadic categories. *Compositionality*, 5(4):56, 2023. arXiv:2105.05198.

[DCV13] Gabriel C Drummond-Cole and Bruno Vallette. The minimal model for the batalin–vilkovisky operad. *Selecta Mathematica*, 19(1):1–47, 2013.

[DOLAPS23] Bérénice Delcroix-Oger, Guillaume Laplante-Anfossi, Vincent Pilaud, and Kurt Stoeckl. Cellular diagonals of permutahedra. *arXiv:2308.12119*, 2023.

[DV21] Malte Dehling and Bruno Vallette. Symmetric homotopy theory for operads. *Algebr. Geom. Topol.*, 21(4):1595–1660, 2021. arXiv:1503.02701.

[Hin97] Vladimir Hinich. Homological algebra of homotopy algebras. *Communications in algebra*, 25(10):3291–3323, 1997.

[KW23] Ralph M. Kaufmann and Benjamin C. Ward. Koszul Feynman categories. *Proc. Amer. Math. Soc.*, 151(8):3253–3267, 2023. arXiv:2108.09251.

Mentioned Sources II

[LA22] Guillaume Laplante-Anfossi. The diagonal of the operahedra. *Adv. Math.*, 405:Paper No. 108494, 50, 2022. arXiv:2110.14062.

[LAM23] Guillaume Laplante-Anfossi and Thibaut Mazuir. The diagonal of the multiplihedra and the tensor product of A_∞-morphisms. *J. Éc. polytech. Math.*, 10:405–446, 2023.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346. Springer Science & Business Media, 2012.

[ML65] Saunders Mac Lane. Categorical algebra. *Bulletin of the American Mathematical Society*, 71(1):40–106, 1965.

[MS06] Martin Markl and Steve Shnider. Associahedra, cellular W-construction and products of A_∞-algebras. *Trans. Amer. Math. Soc.*, 358(6):2353–2372, 2006.

[MTTV21] Naruki Masuda, Hugh Thomas, Andy Tonks, and Bruno Vallette. The diagonal of the associahedra. *J. Éc. polytech. Math.*, 8:121–146, 2021. arXiv:1902.08059.

[Sta63] James Dillon Stasheff. Homotopy associativity of h-spaces. ii. *Transactions of the American Mathematical Society*, 108(2):293–312, 1963.

[Sto24] Kurt Stoeckl. Koszul operads governing props and wheeled props. *Advances in Mathematics*, 454:109869, 2024.

[SU04] Samson Saneblidze and Ronald Umble. Diagonals on the permutahedra, multiplihedra and associahedra. *Homology Homotopy Appl.*, 6(1):363–411, 2004. arXiv:0209109.

[YJ15] Donald Yau and Mark W Johnson. *A foundation for PROPs, algebras, and modules*, volume 203. American Mathematical Soc., 2015.

[Zas75] Thomas Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. *Mem. Amer. Math. Soc.*, 1(issue 1, 154):vii+102, 1975.

Groupoid Coloured Operads

Theorem ([Sto24])

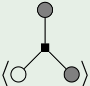
Let P be a groupoid coloured operad such that the associated coloured shuffle operad $(P^f)_*$ admits a quadratic Groebner basis, then P is Koszul.

Let $\mathbb{V}$ be a groupoid, where $Aut(v)$ is finite for all $v \in Ob(\mathbb{V})$.

Example (4.2.14 of [Sto24])

Let $\mathbb{V}$ be the groupoid with three objects a, b, c and a single non-identity isomorphism $f : b \to c$, and its inverse $f^{-1} : c \to b$. Let N be the non-symmetric $ob(\mathbb{V})$-coloured module spanned by a single binary operation

$$N = N\begin{pmatrix} c \\ b, c \end{pmatrix} = \langle \quad \rangle$$



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